ON THE SYMBOLIC CALCULUS OF BERNOULLI CONVOLUTIONS

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ABSTRACT

A probability measure on T whose symbolic calculus is minimal is constucted.

Introduction. Let μ be a probability measure on the circle group T (or the real numbers modulo 1), and let $\hat{\mu}$ be its Fourier-Stielties transform:

$$\hat{\mu}(n) = \int_0^{2\pi} e(-nt)\mu(dt), \qquad n = 0 \pm 1, \pm 2, \cdots,$$

where $e(x) \equiv e^{2\pi i x}$.

If $-1 \leqq \hat{\mu} \leqq 1$, then a function F, continuous in $[-1, 1]$ is said to operate on μ if the composite function $F \circ \hat{\mu}$ is a Fourier-Stieltjes transform of a finite measure on T (not in general a probability). In that case we define $F(\mu)$ by the prescription $\widehat{F(\mu)} \equiv F \circ \hat{\mu}$. The probability μ is called a *Bernoulli convolution* if $\hat{\mu}$ is expressed as a product

$$\hat{\mu}(n) = \prod_{k=1}^{\infty} \cos(2\pi a_k n) \tag{1}$$

where each $a_k \geqq 0$, $\sum_{k=1}^{\infty} a_k^2 < \infty$.

THEOREM. *For a certain Bernoulli convolution μ_0, every function F that operates on μ_0 is a power series*

$$F(x) \equiv \sum_{m=0}^{\infty} b_m x^m, \ \sum_{m=0}^{\infty} |b_m| < \infty.$$

This theorem is related to work of Herz [2], and also to work of Helson, Kahane, Katznelson, and Rudin [1], Varopoulos [5], and the author [4].

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The proof is contained in three paragraphs: the first develops some elementary facts about the internal structure of the family of Bernoulli convolutions; the next contains some Fourier analysis of functions derived from F; and the third is an application of a measure-representation method of Hewitt and Savage [3]. In regard to [3] some references to the necessary facts are given, but it was inexpedient to reproduce the formidable notation.

I. LEMMA. *There exists a Bernoulli convolution μ_0 with this property: For any Bernoulli convolution μ there is a sequence of positive integers $\{A_N\}_{N=1}^{\infty}$ such that*

$$\lim_{N\to\infty} \hat{\mu}_0(A_N n) = \hat{\mu}(n), \qquad n = 0, \pm 1, \pm 2, \cdots.$$

Proof. In the formula (1), it is plainly enough to consider all finite products in which the numbers a_k are rational; let $\{\mu_N\}_{N=1}^{\infty}$ be an enumeration of these. We momentarily apply formula (1) to *real* values of n, and write

$$\hat{\mu}_0(n) = \prod_{N=1}^{\infty} \hat{\mu}_N(nA_N{}^{-1}).$$

Here $A_1, A_2, \cdots$ are integers chosen inductively to meet two requirements:

(i) For $1 \leqq j < N$, and all integers n, $\hat{\mu}_j(nA_NA_j^{-1}) = 1$.

This is plainly possible as all the coefficients involved are rational numbers.

(ii) For $1 \leqq j < N$, $-\infty < u < \infty$,

$$\left|\hat{\mu}_j(uA_jA_N^{-1}) - 1\right| \leqq |u| N^{-2}.$$

This is easily attained from the inequality $|1 - \cos y| \leqq |y|$, $-\infty < y < \infty$. Then $|\hat{\mu}_0(nA_N) - \hat{\mu}_N(n)| \leqq |n| \sum_{j=N} j^{-2}$, and in the statement of the lemma we can choose an appropriate subsequence of the A_N's. Observe that by making the integers A_N increase very rapidly, we can make the support of μ_0 arbitrarily "thin" in terms of Hausdorff dimension.

COROLLARY. *If F operates on μ_0 then F operates on every Bernoulli convolution μ and $\| F(\mu) \| \leqq \| F(\mu_0) \|$.*

This is a consequence of Helly's theorem, the continuity of F, and the lemma.

II. We recall that for a continuous function h on the N-torus T^N, the $A(T^N)$-norm is the sum of the moduli of the Fourier coefficients of h:

$$h^{\wedge}(m_1,\cdots,m_N)=\int_0^1\cdots\int_0^1 e(-m_1x_1-,\cdots,-m_Nx_N)h(x)dx.$$

For each $N \geqq 1$, define

$$H(x_1,\cdots,x_N)=F(\cos(2\pi x_1),\cdots,\cos(2\pi x_N)). \tag{2}$$

LEMMA. *If F operates on μ_0 then for every $N \geqq 1$, $\| H \|_{A(T^N)} \leqq \| F(\mu_0) \|$.*

Proof. Let N be fixed, let $p_1 < p_2 < \cdots < p_N$ be odd primes, and let

$$\hat{\mu}(n) = \prod_{k=1}^{N} \cos(2\pi n p_k^{-1}).$$

By the corollary of I we know that $\| F(\mu) \| \leqq \| F(\mu_0) \|$, and we shall now prove that as $p_1 \to \infty, \cdots, p_N \to \infty$, the norm of $F(\mu)$ converges to $\| H \|_{A(T^N)}$.

The measure $F(\mu)$ is supported by the p-th roots of 1, $p = p_1p_2\cdots p_N$. Each of these roots can be represented modulo 1 in the form

$$w = s_1p_1^{-1} + \cdots + s_Np_N^{-1} \tag{3}$$

where $\frac{1}{2}(1-p_k) < s_k \leqq \frac{1}{2}(1+p_k)$, $1 \leqq k \leqq N$. Different choices of $(s_1,\cdots,s_N)$ yield distinct p-th roots, and each N-tuple $(s_1,\cdots,s_N)$ is actually encountered for large enough primes.

The point-value of $F(\mu)$ at the element w in (3) is

$$p^{-1}\sum_{0<t_K<p_K}\cdots\sum e\left(\sum s_kt_kp_k^{-1}\right)F\left[\prod_{k=1}^{N}\cos(2\pi t\, p_k^{-1})\right].$$

For fixed $(s_1,\cdots,s_N)$ the limit as $p_1 \to \infty,\cdots,p_N \to \infty$, is exactly $\hat{H}(s_1 \cdots s_N)$. But, as explained before, any finite number of these eventually occur separately and simultaneously, so indeed

$$\| H \|_{A(T^N)} \leqq \sup \| F(\mu) \| = \| F(\mu_0) \|.$$

III. LEMMA. *If the $A(T^N)$ norms of H are bounded by B for all $N \geqq 1$ (H being defined by (2)) then*

$$F(x) = \sum_{m=0}^{\infty} b_m x^m, \sum_{m=0}^{\infty} |b_m| \leqq B.$$

The proof of this depends on [3], in particular the main theoorem on p. 483, but some preparation is necessary. First we allow the number of dimensions, N to increase indefinitely, obtaining in the limit a "Fourier expansion" of H as a function of an infinite sequence of co-ordinates. Then we use [3] to represent that expansion as an integral over the elementary, or extreme, measures in a certain compact set of measures. Finally we show that this representation is concentrated on a subset of the extreme measures that corresponds exactly to the functions $1, x, x^2, \cdots$.

(a) Let Z be the set of integers, let $Y = Z \cup \{\phi\}$ be the one-point compactification of Z, and let Z^∞(resp. Y^∞) be the infinite product of Z (resp. Y) with itself. There is a Baire measure λ on Z^∞, such that $\| \lambda \| \leqq B$ and

$$\lambda\{z \in Z^\infty : z_1 = s_1, \cdots, z_N = s_N\} = \hat{H}(s_1, \cdots s_N) \tag{4}$$

for any integers $s_1, \cdots s_N$, $(1 \leqq N < \infty)$. To verify this, we observe first that

$$\hat{H}(s_1, \cdots, s_N) \equiv \sum_{m=-\infty}^{\infty} \hat{H}(s_1, \cdots, s_N\ m)$$

so that all the conditions under (4) are mutually consistent. The hypothesis of the lemma then guarantees that the variation of λ is $\leqq B$. It is perhaps clearer if the measure λ is allowed *a priori* to be supported by the compact space Y^∞, since it follows from (4) that Z^∞ must have full λ-measure.

(b) The measure λ on $Z^\infty \subseteq Y^\infty$ is symmetric [2, p. 472], that is, invariant under finite permutations of the co-ordinates. Then λ is a combination of four symmetric positive measures. To see this, let $\lambda = \lambda_1 + i\lambda_2$ with λ_2 and λ_2 real. Let $\lambda_1 = \lambda_1^+ - \lambda_1^-$ be a Hahn decomposition of λ_1 and S any finite permutation of the co-ordinates. Then

$$\lambda_1 = \lambda_1 \circ S^{-1} = \lambda_1^+ \circ S^{-1} - \lambda_1^- \circ S^{-1}$$

whence $\lambda_1^+ \circ S^{-1} \geqq \lambda_1^+$ and finally $\lambda_1^+ = \lambda_1^+ \circ S^{-1}$. We are now in a position to apply the representation theorem.

(c) Let K be the compact metric space of probability measures in Y^∞, and for each element σ of K, let σ^* be the infinite product of σ with itself so that σ^* is defined in Y^∞. Then there is a Baire measure Φ *in* K such that

$$\langle \lambda, f \rangle = \int_{\sigma \varepsilon K} \langle \sigma^*, f \rangle \Phi(d\sigma)$$

for each continuous function f on Y^∞. Symbolically

$$\lambda = \int_K \sigma^* \Phi(d\sigma).$$

If K is a disjoint union $K_1 \cup K_2$, then the measures

$$\int_{K_1} \sigma^*\Phi(d\sigma) \text{ and } \int_{K_2} \sigma^*\Phi(d\sigma)$$

are mutually singular [3, pp. 487–489]. This proves at once that $\|\Phi\| \leqq B$ and that Φ is uniquely determined.

In the formula for $\langle \lambda, f\rangle$ we can pass to bounded Baire functions on Y^∞. In particular, suppose

$$\|f\| \leqq 1, \quad \langle \lambda, f\rangle = \|\lambda\|, \quad \lambda = 0 \text{ on } Y^\infty \sim Z^\infty.$$

Then $|\langle f, \sigma^*\rangle| = 1$ almost everywhere $d\Phi$, and Φ is concentrated on the subset K_0 of K consisting of measures on Z.

(d) Suppose that $x_1, \cdots, x_N$ are fixed real numbers and set

$$f(z) = e(z_1 x_1 + \cdots + z_N x_N), \; z = (z_1, \cdots, z_N, \cdots).$$

Then $\langle f, \sigma^*\rangle = \hat\sigma(x_1) \cdots \hat\sigma(x_N)$ where $\sigma \equiv \hat\sigma$ is the usual Fourier transformation defined

$$\hat\sigma(x) = \sum_{m=-\infty}^{\infty} \sigma(\{m\}) e(mx).$$

Hence

$$H(x_1, \cdots, x_N) = \int_{K_0} \hat\sigma(x_1) \cdots \hat\sigma(x_N)\Phi(d\sigma).$$

Let $a_1 \; b_1, a_2, b_2$ be real numbers for which

$$\cos(2\pi a_1)\cos(2\pi b_1) = \cos(2\pi a_2)\cos(2\pi b_2). \tag{5}$$

Then $H(a_1, b_1, x_3, \cdots, x_N) \equiv H(a_2, b_2, x_3, \cdots, y_N)$,

and $$\int [\hat\sigma(a_1)\hat\sigma(b_1) - \hat\sigma(a_2)\hat\sigma(b_2)]\hat\sigma(x_3) \cdots \hat\sigma(x_N)\Phi(d\sigma) \equiv 0.$$

The uniqueness mentioned above shows that

(6) $\hat\sigma(a_1)\hat\sigma(b_1) = \hat\sigma(a_2)\hat\sigma(b_2)$ almost everywhere $d\Phi$. We claim

$$(5) \Rightarrow (6), \text{ for } \textit{all} \text{ instances of } (5),$$

almost everywhere. For, in addition to the w^*-topology, K_0 carries a norm topology in which it is a separable metric space, and which yields the same Baire sets as the w^*-topology. But the sets described in (6) are norm-closed, so that the intersection of these sets is in fact a *countable* intersection, and the claim is justified.

It is easily seen that the only transforms $\hat\sigma$ obeying the implication (5) $\Rightarrow$ (6) have the form $\hat\sigma(x) = \cos \kappa\pi x$, $k = 0, 1, 2, \cdots$. This is exactly the form of H asserted in the lemma. The lemma and the main theorem are now proved.

References

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